Topological data analysis for characterizing bone microstructure in medical imaging

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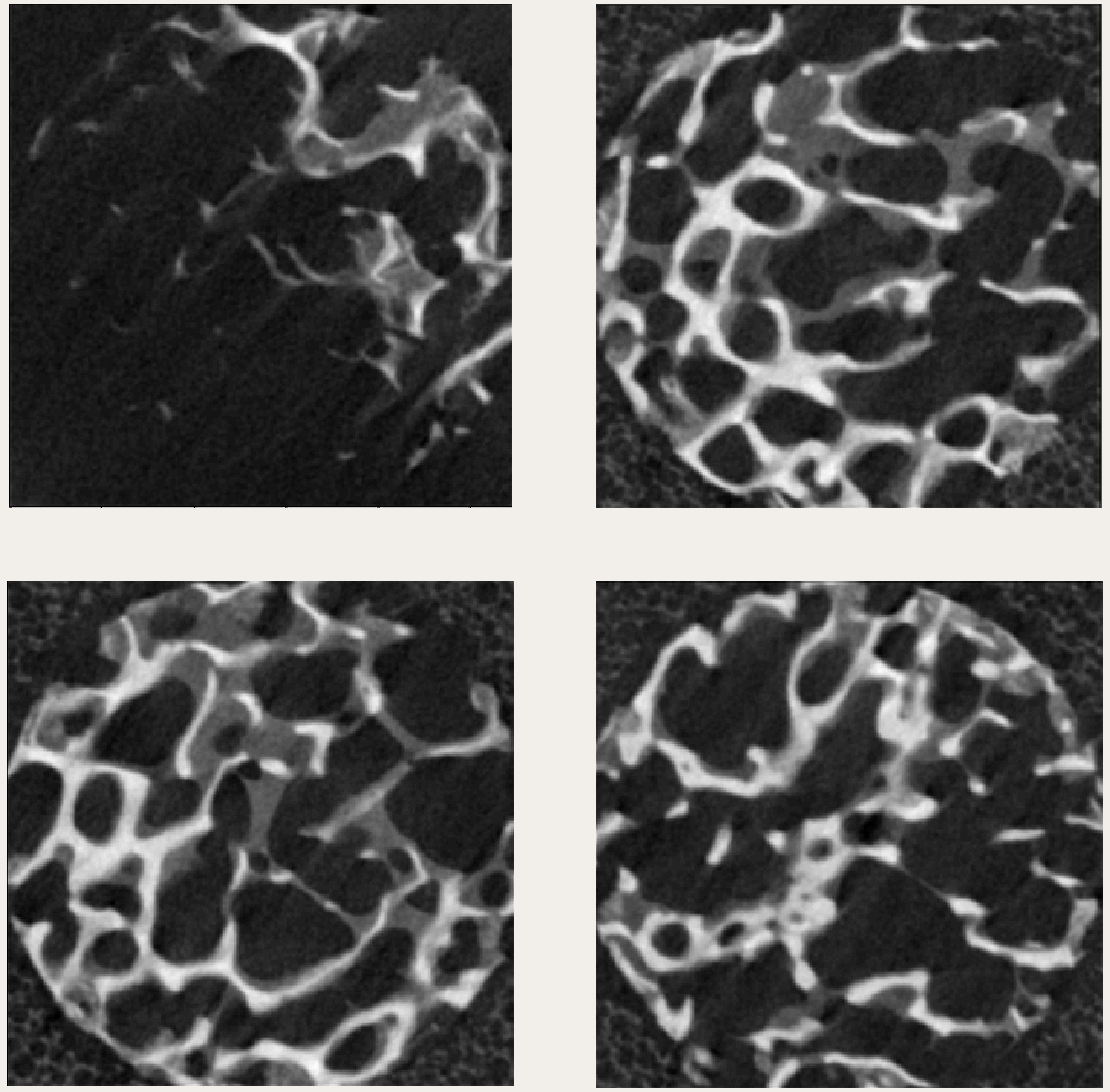
IMAGE DATASET

MICRO-COMPUTED TOMOGRAPHY



$$I : D \subset \mathbb{Z}^3 \to \mathbb{Z}_{\geq 0}$$

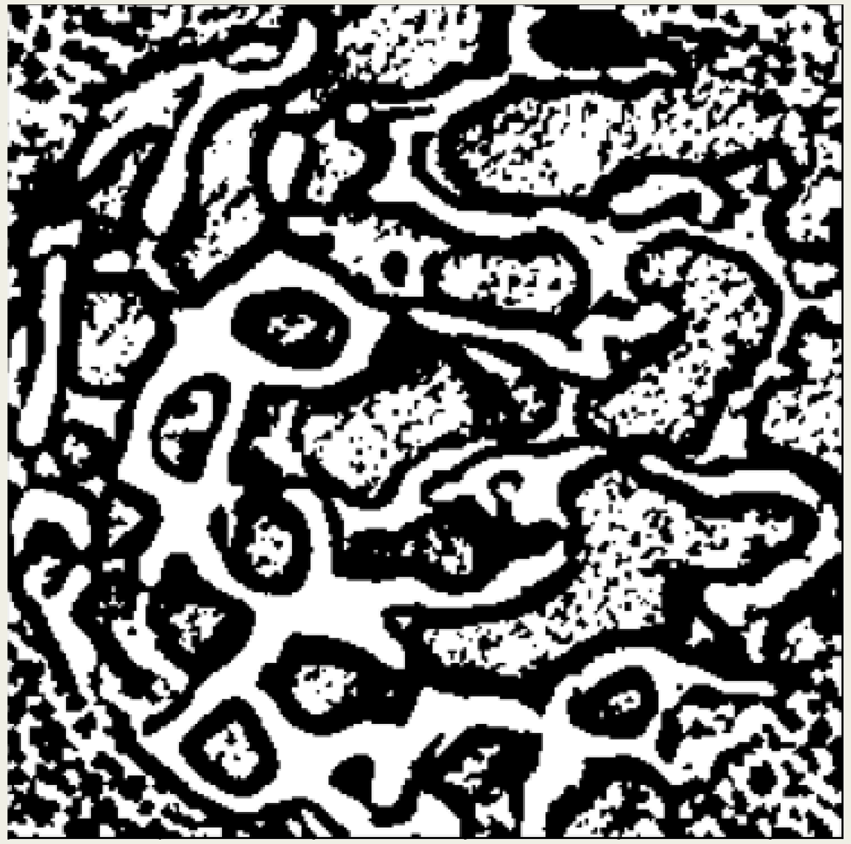
APPARENT STRENGTH

- Bone fragility proxy

$$\mathbb{R}$$

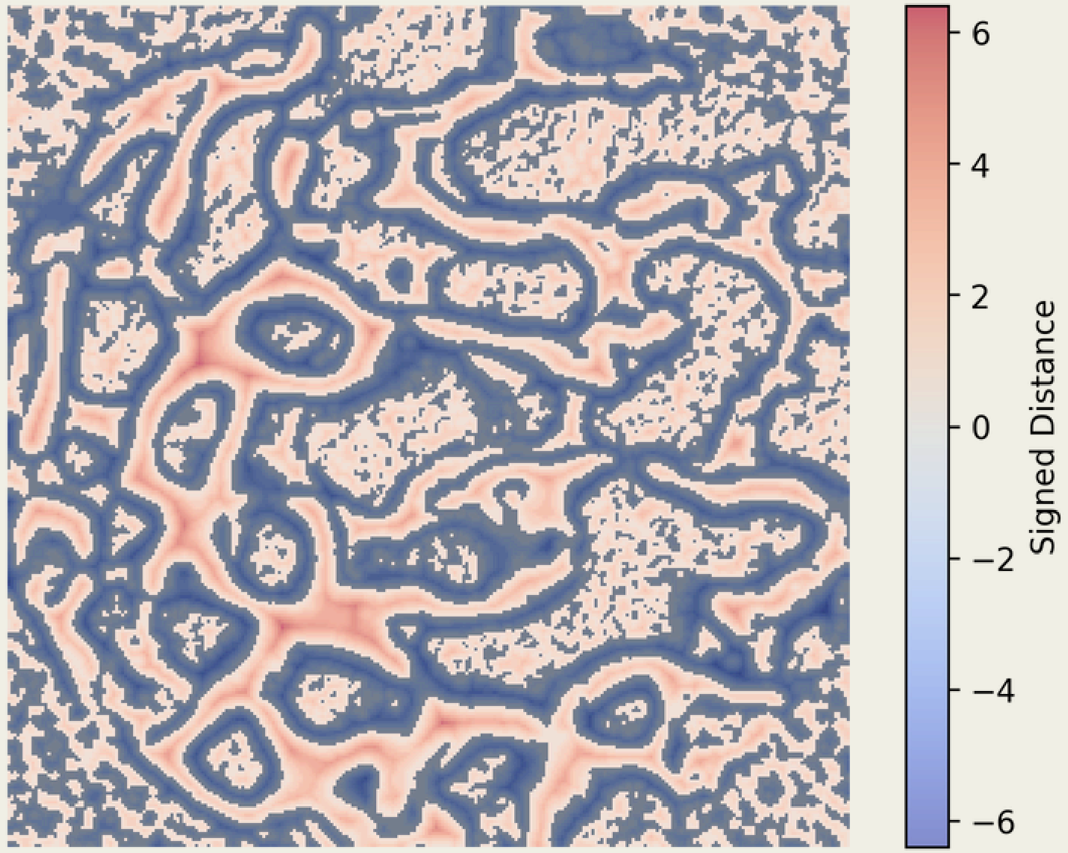
SIGNED DISTANCE TRANSFORM

BINARY IMAGE



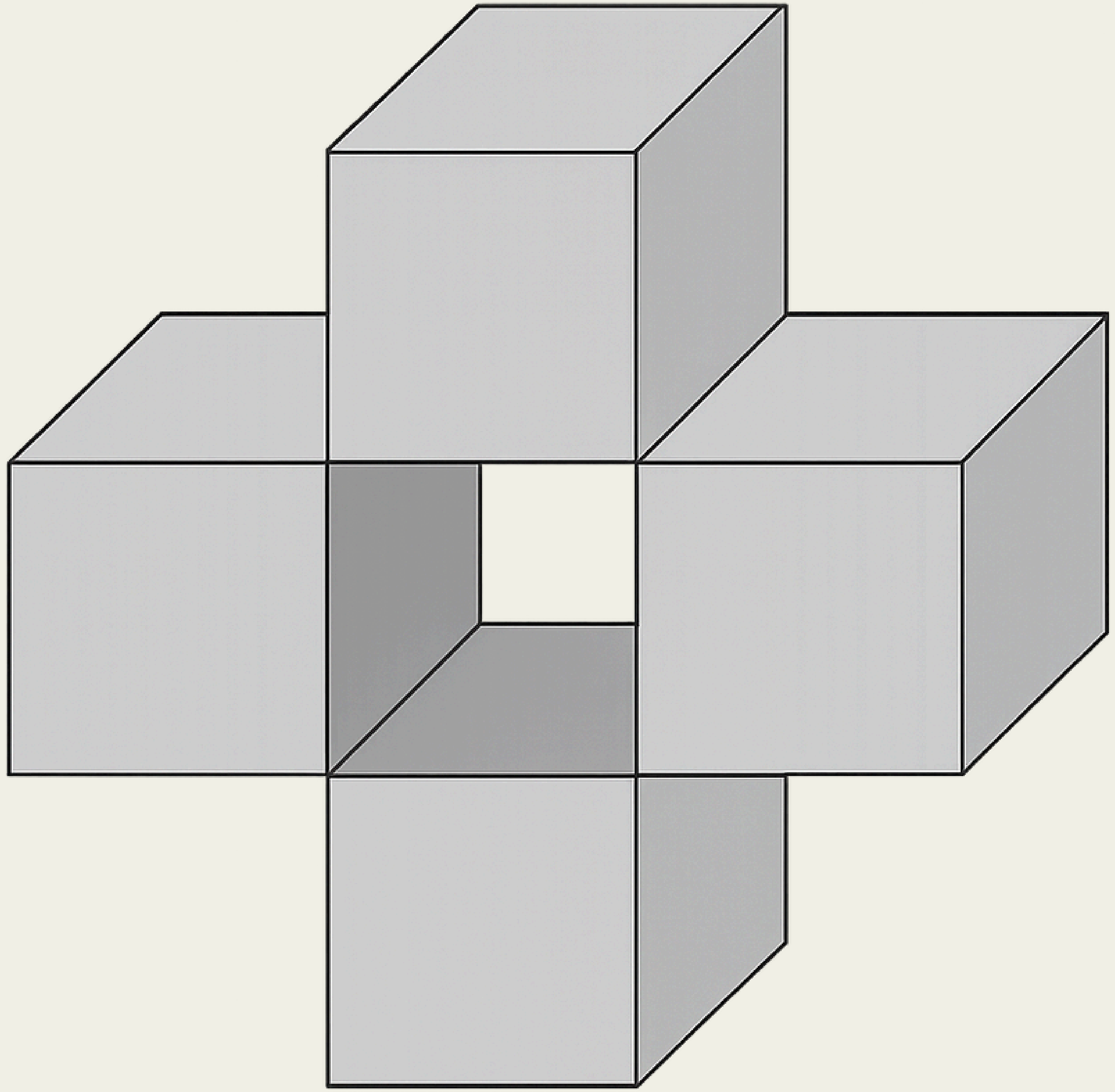
$$B : D \to \{0, 1\}$$

EUCLIDEAN SIGNED DISTANCE



$$S : D \to \mathbb{R}$$

CUBICAL COMPLEX



- Cubical complex K
- Choose some voxels using a sublevel set

$$K_a = \{\sigma \in K \mid f(\sigma) \leq a\}$$

BONE MORPHOMETRY

Sphere Fitting Methods

- Trabecular thickness
- Trabecular spacing

Voxel-Counting Descriptors

- Bone volume
- Bone volume fraction

Topological Descriptors

- Euler number
- Connectivity density

Degree of Anisotropy

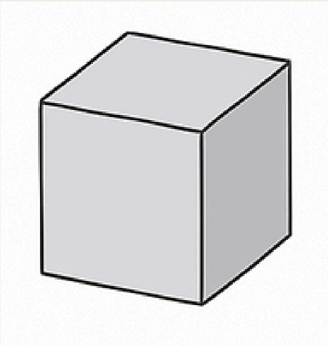
PERSISTENT HOMOLOGY

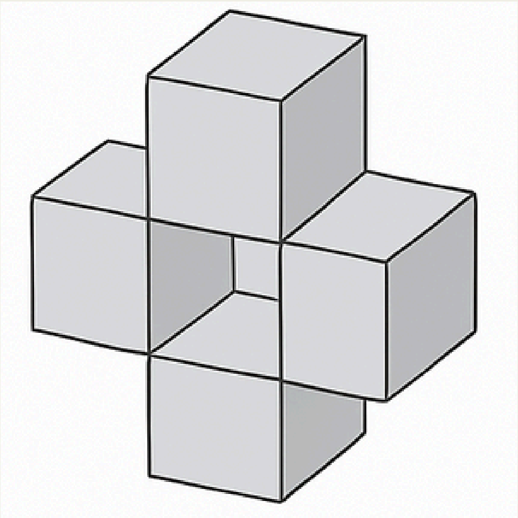
FILTRATION

- For real numbers $a_0 < a_1 < \cdots < a_m$

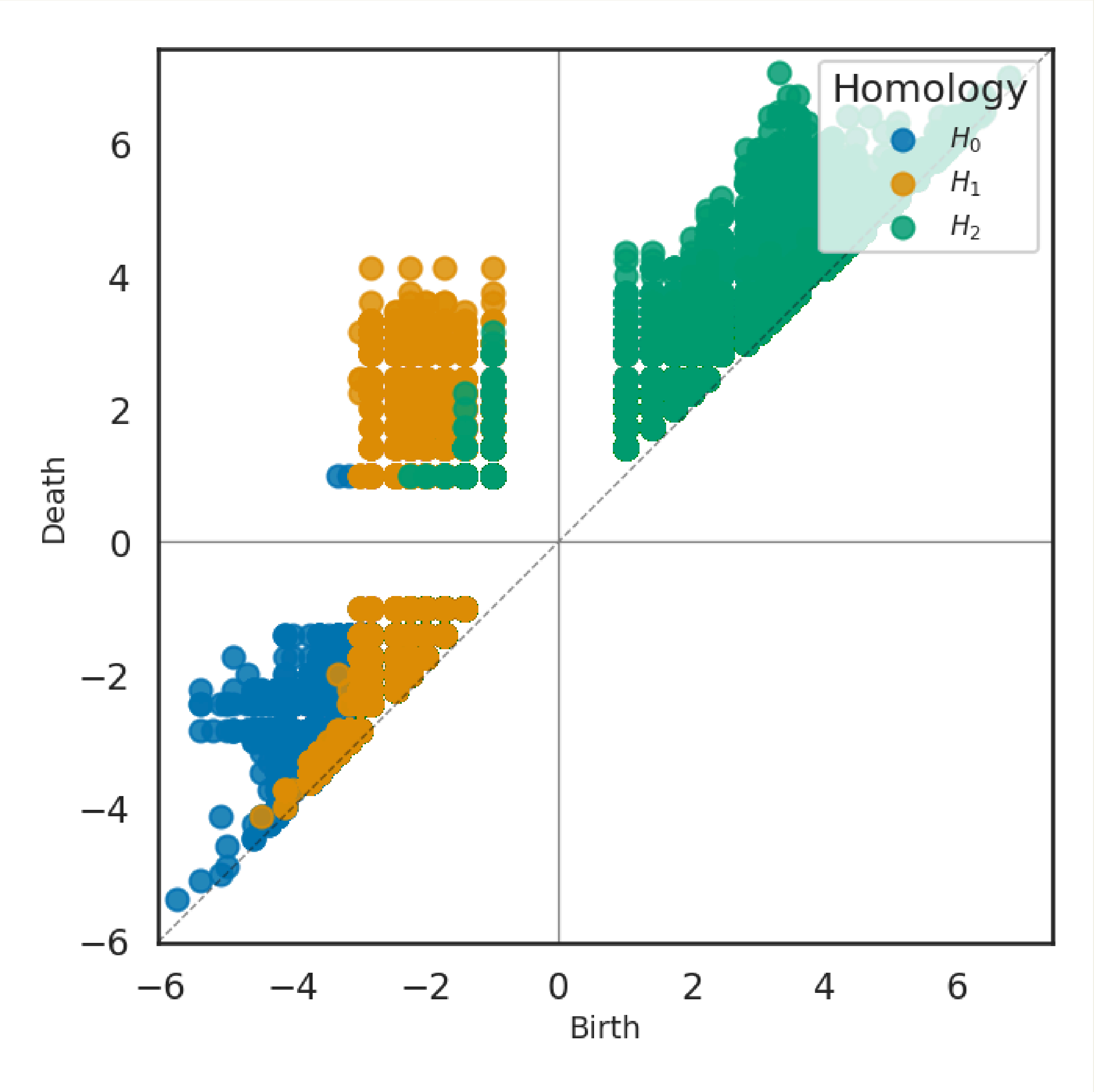
Consider the chain of cubical complexes

$$K_{a_0} \subseteq K_{a_1} \subseteq \cdots \subseteq K_{a_m}$$



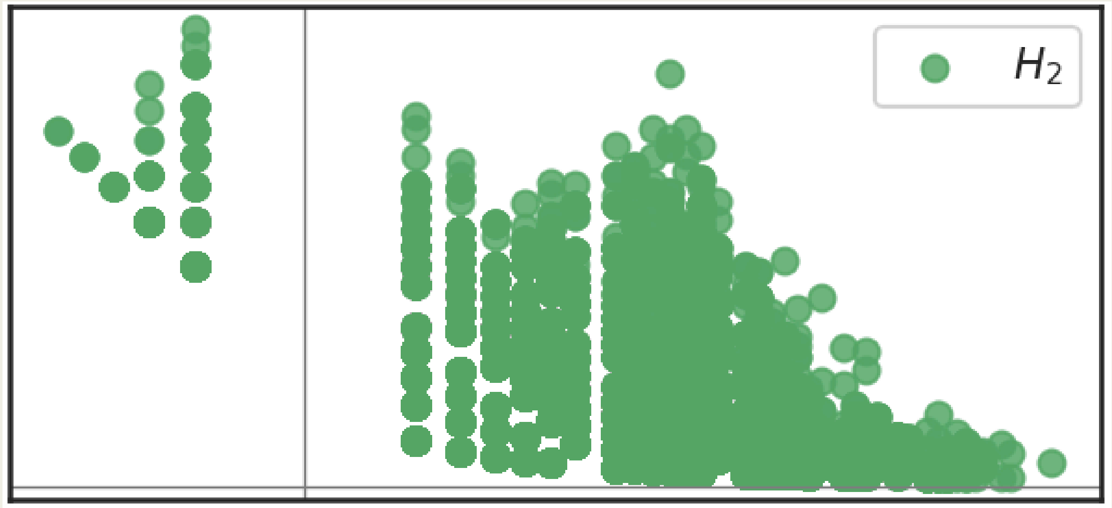


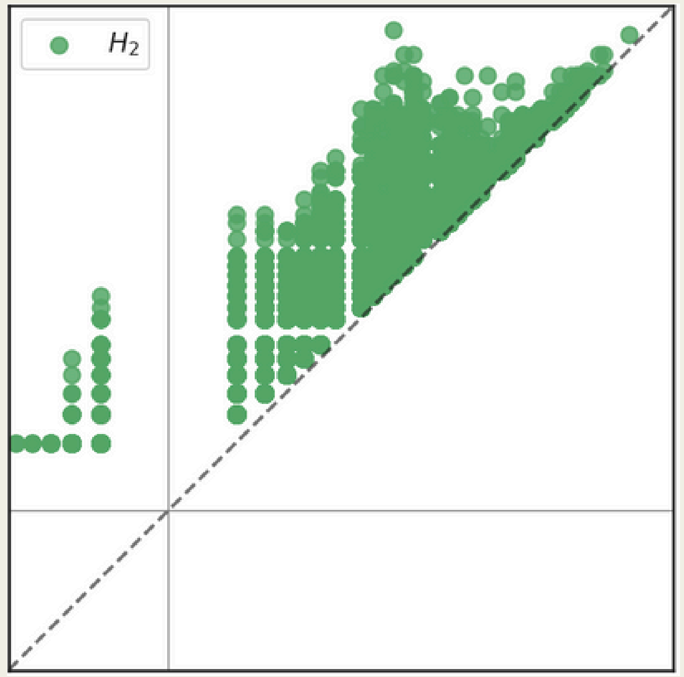
DIAGRAM



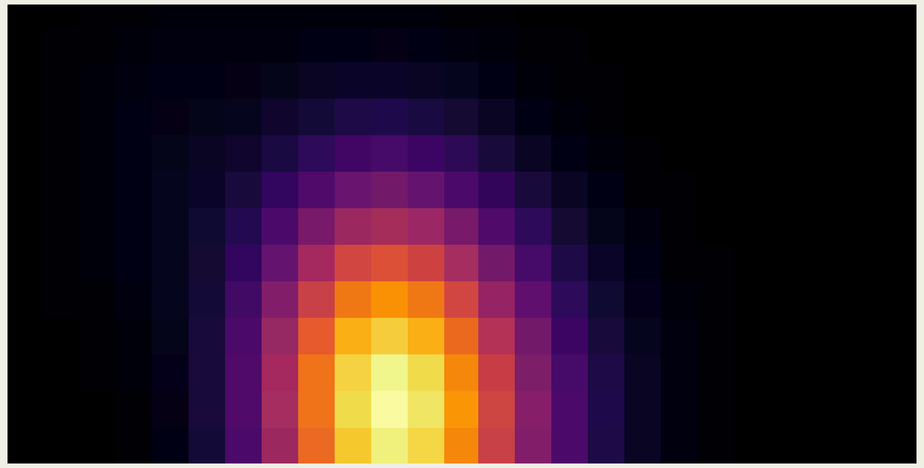
PERSISTENCE IMAGE

BIRTH-PERSISTENCE





DISCRETIZATION



$$\frac{w(b_i, p_i)}{2\pi\sigma^2}\exp\left(-\frac{(x-b_i)^2+(y-d_i)^2}{2\sigma^2}\right)$$

STRENGTH PREDICTION

- Mean of Apparent Strength: 5.52 (±2.35)

BONE MORPHOMETRY

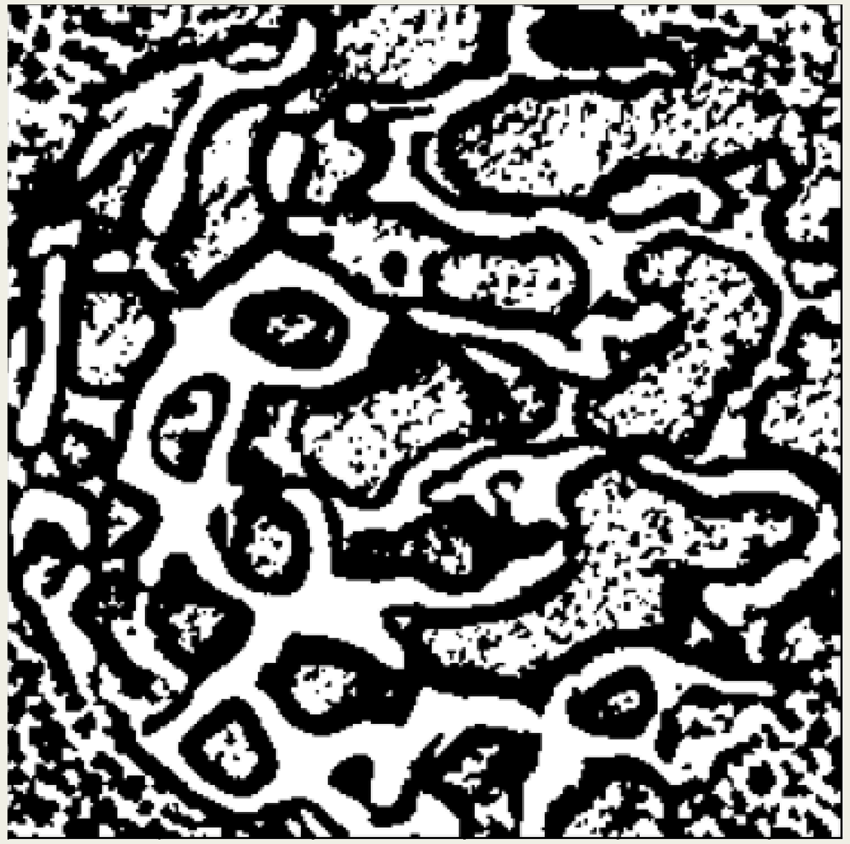
Binary	Model	RMSE	R^2
Otsu	Random Forest	1.78 ± 0.21	0.38 ± 0.11
2D Otsu	Gradient-Boosted Trees (GBT)	**1.48** ± 0.25	**0.56** ± 0.13

PERSISTENCE IMAGE

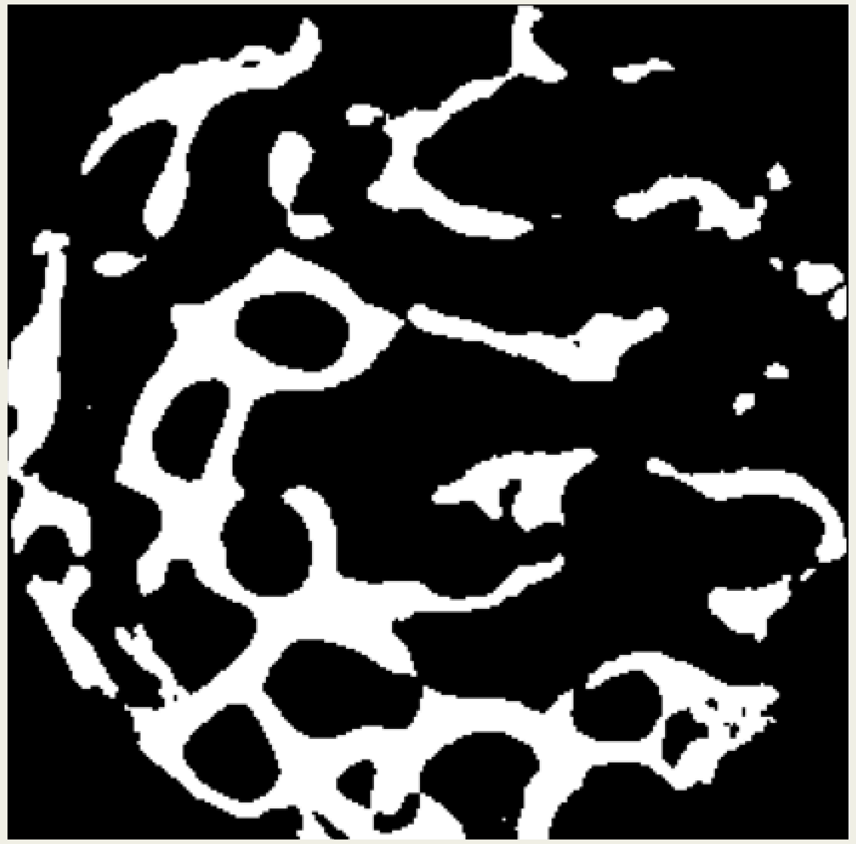
Features	Dimension	Model	RMSE	R^2
PH	0	GBT	1.68 ± 0.26	0.44 ± 0.13
SDPH	2		**0.97** ± 0.29	**0.81** ± 0.09

TAKEAWAY(S)

On binarization methods



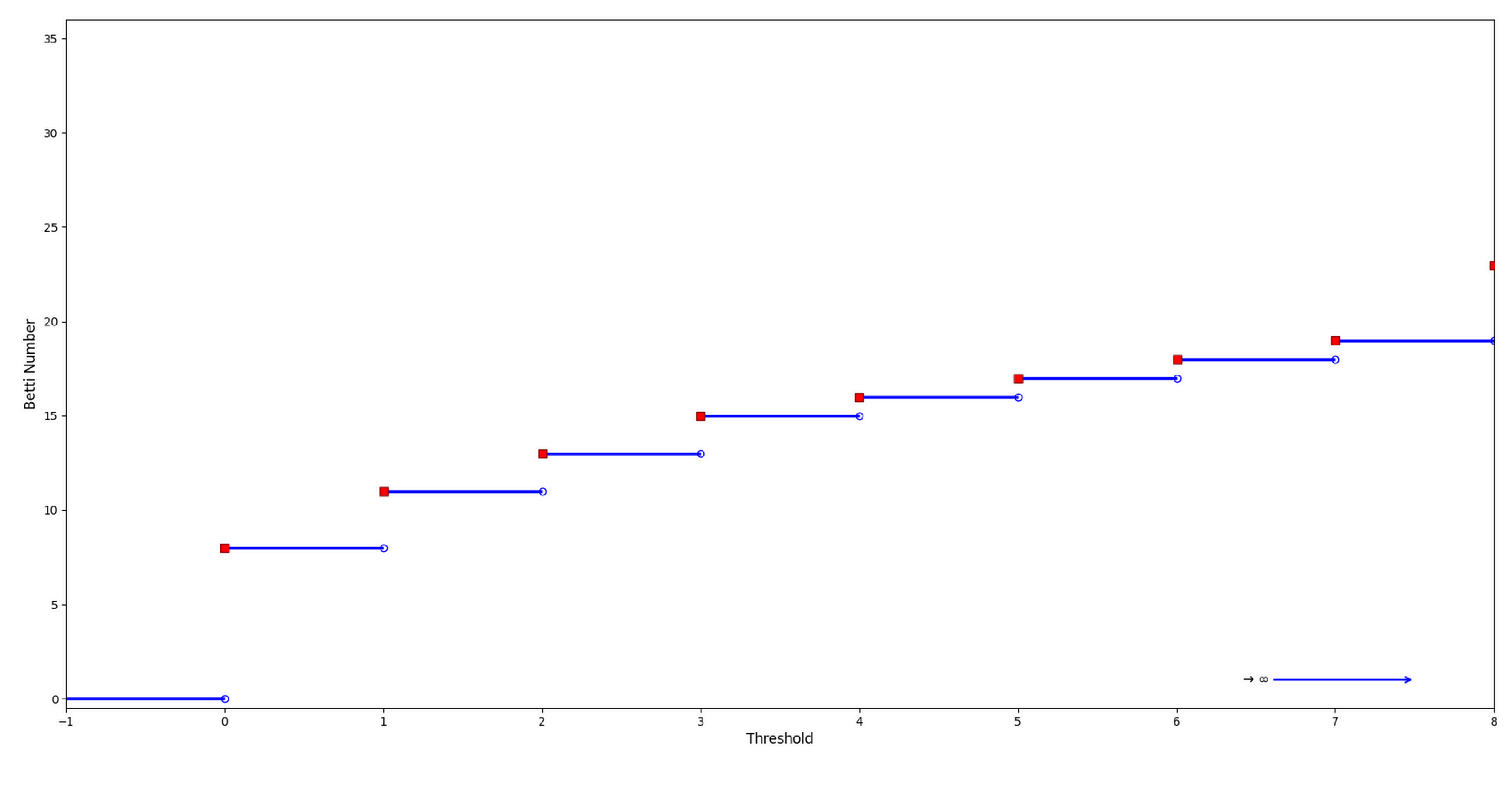
1D Otsu



2D Otsu

BETTI CURVE

ZEROTH BETTI CURVE



MINIMUM SPANNING TREE

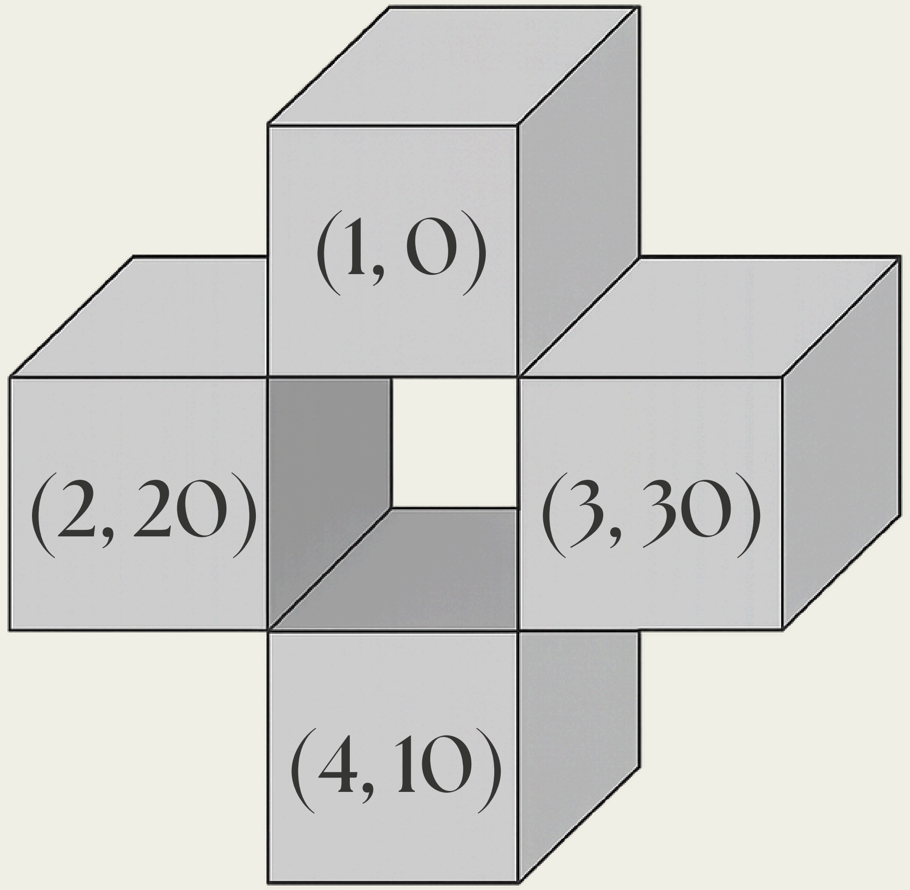
- Weighted graph $G = (V, E, w)$
- Spanning tree $T = (V, E_0, w_0)$
- Minimize

$$w(T) = \sum_{e \in E} w(e)$$

BORUVKA'S ALGORITHM

- Each vertex starts as a connected component
- While there are multiple connected components, do the following:
  - Find edges of **minimum weight** connecting the components.

Neighborhood Graph:
((1, 2), 20)
((1, 3), 30)
((2, 4), 20)
((3, 4), 30)



MST:
((1, 2), 20)
((2, 4), 20)
((1, 3), 30)

SECOND BETTI CURVE

- Complex associated to the integer lattice grid Ω
- The zeroth Betti curve of the filtration

$$\Omega - K_{a_m} \subseteq \cdots \subseteq \Omega - K_{a_1} \subseteq \Omega - K_{a_0}$$

  - 6-neighborhood graph

EULER CHARACTERISTIC CURVE

- For each cube, we compute the change in the Euler characteristic.

$$\Delta\chi(t) = \#(N_0) - \#(N_1) + \#(N_2) - 1$$

- First Betti numbers can now be computed.

$$\chi(t) = \beta_0(t) - \beta_1(t) + \beta_2(t)$$

DISTRIBUTED COMPUTING

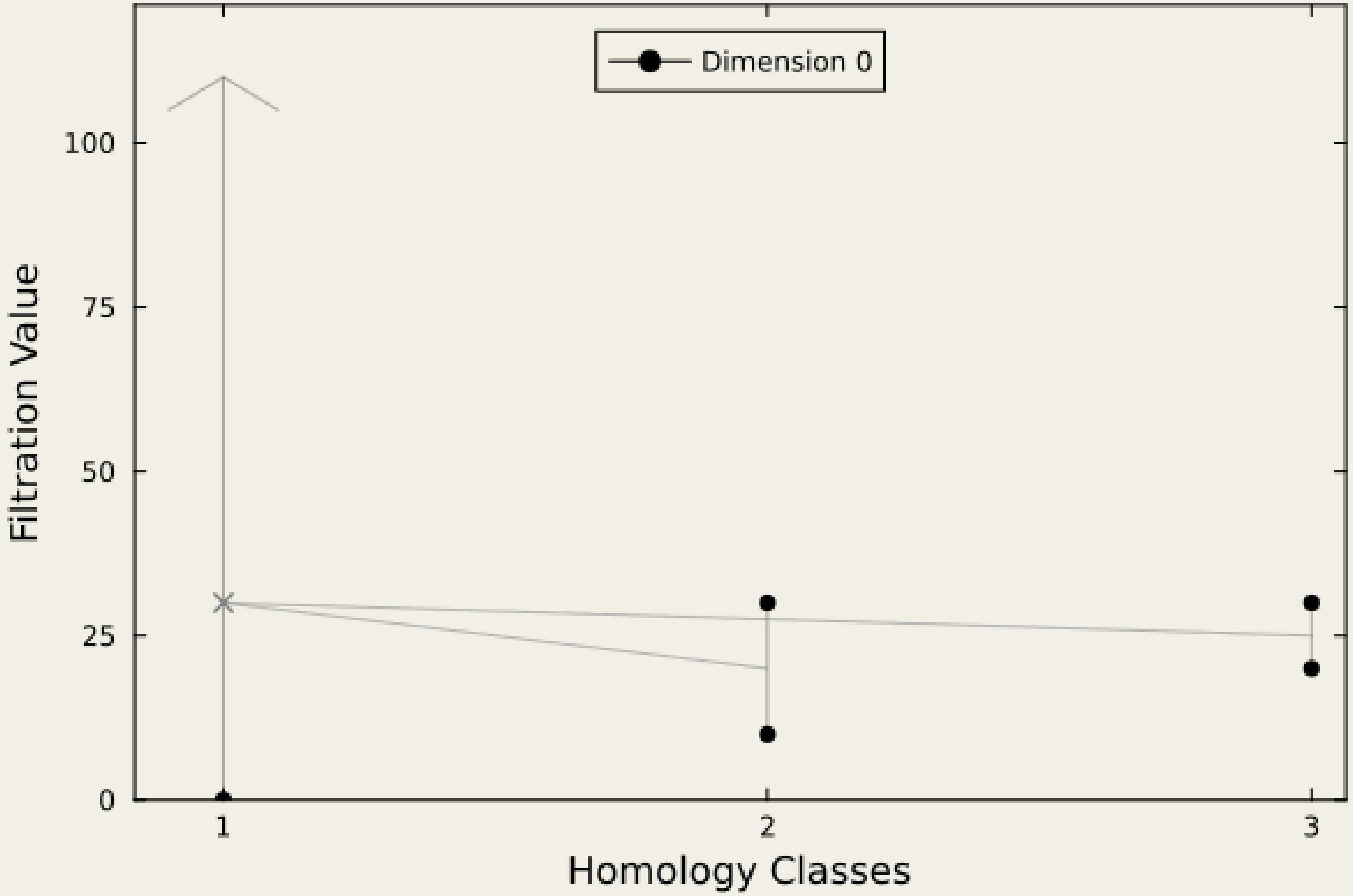
- Perform the operations on **image chunks** or subvolumes
  - Store neighborhood graph (in a distributed way)
  - Compute Betti curves based from graph

Shape	**Number of Voxels**	**Runtime** (seconds)
10 x 10 x 10	1000	0.457
20 x 20 x 20	8000	0.81
40 x 40 x 40	64000	2.014
100 x 100 x 100	1000000	29.889
250 x 250 x 250	15625000	447.031

GENERALIZED MERGE TREE

MERGE TREE

- Merging of connected components



HOMOLOGY CLASSES

- Homology group

$$H_n(X) = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})}$$

- Merging of homology classes

$$[z_m] = \sum_{i=1}^{m-1} \alpha_i [z_i] + \partial \rho$$

MERGE FOREST